7) Expectation and Variance

The expectation of a random variable can be thought as the centre of mass of the probability distribution.

The expected value also called expection or mean gives the centre - in the sense the average value of the distribution

The variance is the measure of spread of the random variable.

Defn 7.1: If X is a discrete random variable, the expectation of X (or expected value of X) denoted by $E[X]$ is defined by

$$\boxed{E[X] = \sum_{x \in X(\Omega)} x\, p_X(x) = \sum_{x \in X(\Omega)} x\, P(X=x)}$$

if this series is absolutely convergent.

If not, expection is not defined.

Example: 7.2 **Expectation of Bernoulli Distribution:**

If $X \sim Ber(p)$ then $X(\Omega) = \{0,1\}$

$$E[X] = \sum_{x \in \{0,1\}} x\, p_X(x)$$

$$= 0.(1-p) + 1.p$$

$$= p$$

$$\Rightarrow \quad E[X] = p \quad \text{for } X \sim Ber(p)$$

Example: 7.3 **Expectation of Geometric distribution:**

If $X \sim Geo(p)$ then $X(\Omega) = \{1,2,3,\dots\}$
writing $q = 1-p$,

$$E[X] = \sum_{k=1}^{\infty} k q^{k-1} p$$

$$= p \sum_{k=1}^{\infty} k q^{k-1} + 0$$

$k=0 \quad (k-1)=(0-1)$

$$= p\sum_{k=1}^{\infty} k q^{k-1} + 0 \cdot p q^{0-1}$$

$$= p\left[\sum_{k=1}^{\infty} k q^{k-1} + 0 \cdot p q^{0-1}\right]$$

$$= p\sum_{k=0}^{\infty} k q^{k-1}$$

$$= p\sum_{k=0}^{\infty} \frac{d}{dq} q^{k}$$

$\left[\text{Since } \frac{d}{dq} q^{k} = k q^{k-1}\right]$

$$= p\frac{d}{dq}\left[\sum_{k=0}^{\infty} q^{k}\right]$$

[sum of derivative is derivative of sum]

$$= p\frac{d}{dq}\left[\frac{1}{1-q}\right]$$

formula for infinite geometric sum

$\sum_{x=0}^{\infty} x^{n} = \frac{1}{1-x}$ where $|x| \leq 1$

$$= p\left(\frac{1}{(1-q)^2}\right)$$

(by chain rule)

$$= \frac{p}{p^2}$$

$$= \frac{1}{p}$$

Therefore for $\boxed{X \sim Geo(p)}$

$$\boxed{E[X] = \frac{1}{p}}$$

Example: **Expectation for Poisson Distribution:**
7.4

If $X \sim Pois(\lambda)$ then

$$E[X] = \sum_{k=0}^{\infty} k\, p_X(k)$$

$$= \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda}$$

$$= e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!}$$

$$\left[\frac{k}{k!} = \frac{k}{k(k-1)!} = \frac{1}{(k-1)!} \right]$$

$$= e^{-\lambda} \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}$$

$$= e^{-\lambda} \lambda \sum_{j=0}^{\infty} \frac{\lambda^{j}}{j!}$$

change of variable
let $j = k-1$
$k=1 \Rightarrow k-1=0 \Rightarrow j=0$

taylor series for exponential fn

$$= e^{-\lambda} \lambda e^{\lambda}$$

$$= \lambda.$$

Therefore for $\boxed{X \sim \text{Pois}(\lambda)}$

$$\boxed{E[X] = \lambda}$$

Example: 7.5 Consider the following:
You throw a fair die and

- lose £1 if 1,2 or 3 comes up
- gain nothing (£0) if 4 comes up
- win £1 if 5 comes up
- win £2 if 6 comes up

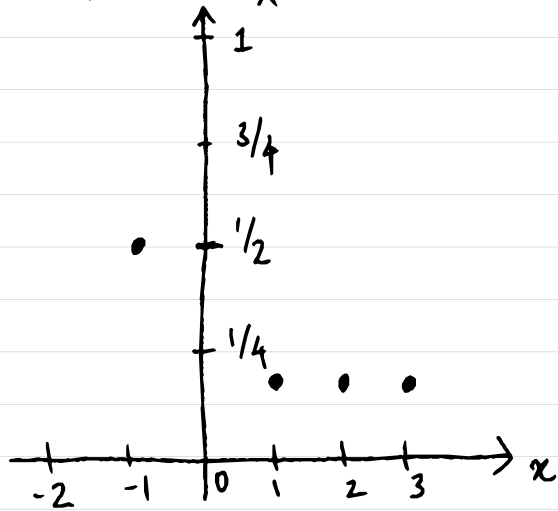
These winnings are encoded in random variable X with range $X(\Omega) = \{-1, 0, 1, 2\}$ defined by

$$X(w) = \begin{cases} -1 & \text{if } w \in \{1,2,3\} \\ 0 & \text{if } w = 4 \\ 1 & \text{if } w = 5 \\ 2 & \text{if } w = 6 \end{cases}$$

The probability mass function is

$$p_X(x) = P(X = x) = \begin{cases} 1/2 & \text{if } x = -1 \\ 1/6 & \text{if } x \in \{0,1,2\} \\ 0 & \text{if } x \notin \{-1,0,1,2\} \end{cases}$$

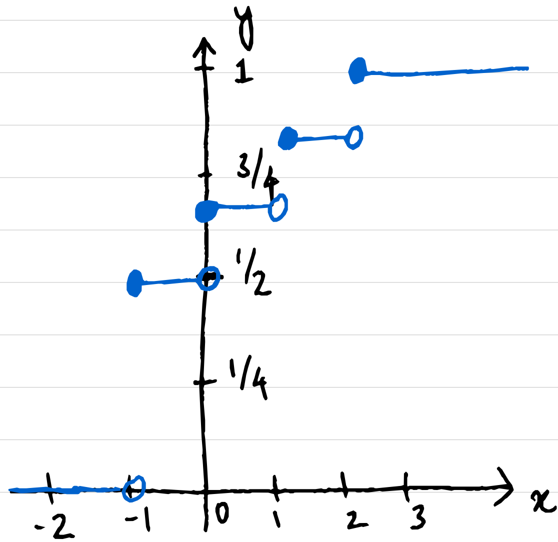
Graph of $p_X(x)$



The distribution function $F_X(x)$ is

$$F_X(x) = \begin{cases} 0 & \text{if } x < -1 \\ 1/2 & \text{if } -1 \leq x < 0 \\ 2/3 & \text{if } 0 \leq x < 1 \\ 5/6 & \text{if } 1 \leq x < 2 \\ 1 & \text{if } 2 \leq x \end{cases}$$

Graph of $F_X(x)$



Calculating the expected gain:

$$E[X] = \sum_{x \in \{-1,0,1,2\}} x \cdot p_X(x)$$

$$= -1 \cdot \frac{1}{2} + 0 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} = 0$$

$\Rightarrow \; E[X] = 0$

Now assume that the government imposes 50% tax on all gambling transactions, so that the tax income is given by the variable

$$T = \frac{1}{2}|X|$$

<u>Calculations for</u>: $T = \frac{1}{2}|X|$

(let $h(x) = \frac{1}{2}|x|$)

$$T = \begin{cases} 1/2 & \text{if } X(\omega) = -1 \\ 0 & \text{if } X(\omega) = 0 \\ 1/2 & \text{if } X(\omega) = 1 \\ 1 & \text{if } X(\omega) = 2 \end{cases} \quad \begin{array}{l} \frac{1}{2}|X| \\ \longrightarrow \frac{1}{2}|-1| = 1/2 \\ \longrightarrow 1/2|0| = 0 \\ \longrightarrow 1/2|1| = 1/2 \\ \longrightarrow 1/2|2| = 1 \end{array}$$

Grouping together common terms gives

$$T = \begin{cases} 1/2 & \text{if } X \in \{-1, 1\} \\ 0 & \text{if } X = 0 \\ 1 & \text{if } X = 2 \end{cases}$$

$$p_T(t) = P(T=t) = \begin{cases} 4/6 = 2/3 & \text{if } t = 1/2 \\ 1/6 & \text{if } t = 1 \\ 1/6 & \text{if } t = 0 \\ 0 & \text{if } t \notin \{0, 1/2, 1\} \end{cases}$$

preimage

$$P(T=1/2) = P(\{T=1/2\})$$

[sum of so all the $x \in X(\Omega)$ s.t $h(x) = t$]

[here $P(T=t) = P(h(X)=t)$] $= P(\{X=-1\} \cup \{X=1\})$

[since random variables partition Ω, apply (P3)] $= P(X=-1) + P(X=1) = \sum_{\substack{x \in X(\Omega) \\ h(x)=t \\ =1/2}} p_X(x)$

$$= 1/2 + 1/6 = 4/6$$

Similarly

$$P(T=1) = P(\{T=1\})$$

$$= P(\{X=2\}) = 1/6$$

$$\sum_{\substack{x \in X(\Omega) \\ h(x)=t}} p_X(x) = 1/6$$

$$P(T=0) = P(\{T=0\})$$

$$= P(X=0) = 1/6$$

The distribution function is

$$F_T(t) = P(T \leq t) = \begin{cases} 0 & \text{if } t < 0 \\ 1/6 & \text{if } t \leq 0 < 1/2 \\ 5/6 & \text{if } 1/2 \leq t < 1 \\ 1 & \text{if } t \geq 1 \end{cases}$$

The expected tax income (expected value):

$$E[T] = 0 \cdot 1/6 + \frac{1}{2} \cdot \frac{2}{3} + 1 \cdot \frac{1}{6} = \frac{1}{2}$$

<u>Generalization of example 7.5</u>:

If $X : \Omega \to \mathbb{R}$ is a discrete random variable
and $h : \mathbb{R} \to \mathbb{R}$ is a function,
Then we would like to compute $E[h(X)]$

In example 7.5,

$$h(X) = 1/2 \, |X|$$

Theorem 7.6: If X is a discrete random variable and $h: \mathbb{R} \to \mathbb{R}$ a function so that

$$Y = h(X) = h \circ X$$

composition

then

$$\boxed{E[h(X)] = \sum_{x \in X(\Omega)} h(x) p_X(x)}$$

if this series is absolutely convergent.

proof: We have by defn 7.1 that

$$E[h(X)] = \sum_{y \in h(X(\Omega))} y P(h(X) = y)$$

The probability of $h(X) = y$ is the sum over all the possible values of X that get mapped to y by h

$$P(h(X) = y) = \sum_{\substack{x \in X(\Omega) \\ h(x) = y}} p_X(x) = P(X = x)$$

Thus

$$E[Y] = \sum_{y \in h(X(\Omega))} y \cdot P(h(X) = y)$$

$$= \sum_{y \in h(X(\Omega))} y \cdot \sum_{\substack{x \in X(\Omega) \\ h(x) = y}} P(X = x)$$

$(y = h(x))$

$$= \sum_{y \in h(X(\Omega))} \sum_{\substack{x \in X(\Omega) \\ h(x) = y}} h(x) P(X = x)$$

(by summation laws)
(since $y = h(x)$ for $x \in X(\Omega)$)

$$= \sum_{x \in X(\Omega)} h(x) p_X(x)$$

■

Example 7.7: Suppose that $X \sim Pois(\lambda)$

We want to find expectation of

$$h(X) = Y = e^X$$

Taking $h(x) = e^x$

$$E[e^X] = \sum_{k=0}^{\infty} e^k \, p_X(x)$$

$$= \sum_{k=0}^{\infty} e^k \frac{\lambda^k}{k!} e^{-\lambda}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(e\lambda)^k}{k!}$$

$= \frac{x^k}{k!}$ for $x = e\lambda$

$=$ exponential fn taylor series

$$= e^{-\lambda} e^{e\lambda}$$

$$= e^{\lambda(e-1)}$$

7.2 Expectation of Continuous random Variables

Defn 7.8: If X is a continuous random variable with density function f_X then the expectation of X denoted by $E[X]$ is defined as

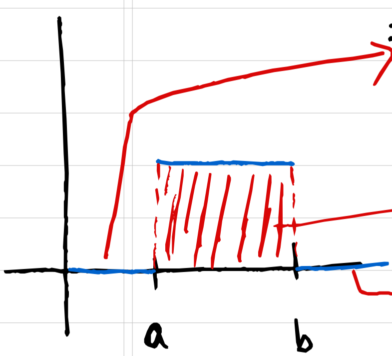
$$\boxed{E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx}$$

whenever the integral is absolutely convergent.

Example 7.9: If $X \sim U(a, b)$ then

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$= \int_{-\infty}^{a} x f_X(x)\, dx + \int_{a}^{b} x f_X(x)\, dx + \int_{b}^{\infty} x f_X(x)\, dx$$



$$= 0 + \int_a^b \frac{x}{b-a}\,dx + 0$$

$$= \frac{1}{b-a}\int_a^b x\,dx$$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b = \frac{1}{b-a}\left[\frac{b^2}{2} - \frac{a^2}{2}\right]$$

$$= \frac{1}{\cancel{b-a}} \cdot \frac{(b+a)\cancel{(b-a)}}{2}$$

$$= \frac{b+a}{2}$$

Therefore for $\boxed{X \sim U(a,b)}$

$$\boxed{E[X] = \frac{b+a}{2}}$$

Example: If $X \sim N(\mu, \sigma^2)$ then
7.10

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} x e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

We use change of variable (integration by substitution)

$$z = \frac{x-\mu}{\sigma} \Rightarrow \sigma\, dz = dx$$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} x e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}\,\cancel{\sigma}} \int_{-\infty}^{\infty} (\sigma z + \mu) e^{-\frac{1}{2}z^2} \cancel{\sigma}.\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z + \mu) e^{-1/2 z^2} dz$$

$$= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{-\frac{1}{2}z^2} dz + \mu \boxed{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz}$$

this integral is 1 by property (d2), since the integrand is the density function of $N(0,1)$ random variable

$$= \frac{-\sigma}{\sqrt{2\pi}} \left[e^{-\frac{1}{2}z^2} \right]_{-\infty}^{\infty} + \mu$$

$$= 0 + \mu = \boxed{\mu}$$

Therefore for $\boxed{X \sim N(\mu, \sigma^2)}$,

$$\boxed{E[X] = \mu}$$

Example: The expectation of exponential distribution $\boxed{X \sim Exp(\lambda)}$ can be calculated as

$$\boxed{E[X] = \int_{-\infty}^{\infty} x e^{-\lambda x} = \frac{1}{\lambda}}$$

If X is a continuous random variable and $h: \mathbb{R} \to \mathbb{R}$ is a (Borel) function, such that $h(X)$ is integrable, we then have a formula for $E[h(X)]$

Given in Theorem 7.11

Theorem 7.11: If X is a continuous random variable with density function f_X and $h: \mathbb{R} \to \mathbb{R}$ is a function then

$$\boxed{E[h(X)] = \int_{-\infty}^{\infty} h(x) f_X(x)\, dx}$$

Example 7.12: If $X \sim U(0,1)$, then letting $h(x) = 1/(x+1)$

$$E\left[\frac{1}{X+1}\right] = \int_0^1 \frac{1}{x+1}\, dx = \Big[\log(x+1)\Big]_0^1$$

$$= \log 2$$

Defn 7.13 The m-th moment of a random variable X is the value $E[X^m]$

Example 7.14 Let $X \sim Exp(\lambda)$ Then

$$E[X^m] = \frac{m!}{\lambda^m} \quad \text{for all } m \in \mathbb{N} \cup \{0\}$$

proof by induction:

Base case $m=0$:
Showing that statement is true for $m=0$:

$$E[X^0] = E[1] = 1 = \frac{0!}{\lambda^0}$$

Inductive hypothesis:
Assume that statement holds for some $k \in \mathbb{N} \cup \{0\}$

Assume that

$$E[X^k] = \frac{k!}{\lambda^k}$$

$$E[x^k] = \int_{-\infty}^{\infty} x^k \lambda e^{-\lambda x} dx = \int_{0}^{\infty} x^k \lambda e^{-\lambda x} dx$$

Inductive step:

Showing that $\forall k \in \mathbb{N} \cup \{0\}$ if the property holds for some $n=k$, then it holds for $n=k+1$,

$$E[X^{k+1}] = \int_{-\infty}^{\infty} x^{k+1} f_X(x)\,dx$$

$$= \int_{-\infty}^{0} x^{k+1} f_X(x)\,dx + \int_{0}^{\infty} x^{k+1} f_X(x)\,dx$$

$$= 0 + \int_{0}^{\infty} x^{k+1} f_X(x)\,dx$$

$$= \int_{0}^{\infty} x^{k+1} \lambda e^{-\lambda x}\,dx$$

$$= \int_{0}^{\infty} x^{k+1} \frac{d}{dx}\left(-e^{-\lambda x}\right) dx$$

$\left[\text{since } \frac{d}{dx}\left(-e^{-\lambda x}\right) = \lambda e^{-\lambda x}\right]$

$$= -\left[x^{k+1}e^{-\lambda x}\right]_0^{\infty} + \int_0^{\infty}(k+1)x^k e^{-\lambda x}\,dx$$

[by applying integration by parts]

$$= 0 + \frac{k+1}{\lambda}\int_0^{\infty} x^k \lambda e^{-\lambda x}\,dx$$

by inductive hypothesis

$$= 0 + \frac{k+1}{\lambda}.E[x^k]$$

$$= \left(\frac{k+1}{\lambda}\right)\cdot\left(\frac{k!}{\lambda^k}\right)$$

$$= \frac{(k+1)!}{\lambda^{k+1}}$$

Hence the property is true for all $m \in \mathbb{N} \cup \{0\}$ by induction. ∎

In particular, $\boxed{E[X] = 1/\lambda}$ (special case $m=1$ $X \sim Exp(\lambda$)

The next example shows that expectation is not defined for all random variables.

Example: 7.15 Let $X \sim Par(\alpha)$. Then

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$= \int_{-\infty}^{1} x f_X(x)\, dx + \int_{1}^{\infty} x f_X(x)\, dx$$

$$= 0 + \int_{1}^{\infty} x f_X(x)\, dx$$

$$= \int_{1}^{\infty} x \frac{\alpha}{x^{\alpha+1}}\, dx$$

$$= \int_{1}^{\infty} \alpha x^{-\alpha}\, dx$$

If $\alpha = 1$, then this formula is not applicable as

$$E[X] = \int_1^{\infty} \frac{1}{x} dx = [\log x]_1^{\infty} = \infty$$

So the expection is undefined

If $\alpha \neq 1$ then the formula gives

$$E[X] = \alpha \int_1^{\infty} x^{-\alpha} dx = \frac{\alpha}{1-\alpha} \left[x^{1-\alpha} \right]_1^{\infty} \quad (*)$$

We see that when $\alpha < 1$, $(*)$ does not converge, and expection is undefined

However when $\alpha > 1$, then

$$\boxed{E[X] = \frac{\alpha}{\alpha - 1}}$$

So in pareto distribution expection only defined when $\boxed{\alpha > 1}$

Theorem 7.16: (Linearity of Expectations)

Let X be a random variable. Then for any $a, b \in \mathbb{R}$

$$\boxed{E[aX+b] = aE[X] + b}$$

proof: Case 1: X is a continuous random variable.
We can use theorem 7.11 with $h(x) = ax+b$

$$E[aX+b] = \int_{-\infty}^{\infty} (ax+b) f_X(x)\, dx$$

$$= a\underbrace{\int_{-\infty}^{\infty} x f_X(x)\, dx}_{= E[X] \text{ by defn}} + b\underbrace{\int_{-\infty}^{\infty} f_X(x)\, dx}_{=1 \text{ by } (d2)}$$

$$= aE[X] + b$$

Case 2: X is a discrete random variable.
We can use theorem 7.6 with $h(x) = ax + b$

$$E[aX+b] = \sum_{x \in X(\Omega)} (ax+b) P_X(x)$$

$$= a \underbrace{\boxed{\sum_{x \in X(\Omega)} x P_X(x)}}_{= E[X] \text{ by defn}} + b \underbrace{\boxed{\sum_{x \in X(\Omega)} P_X(x)}}_{= 1 \text{ by (m2)}}$$

$$= aE[X] + b$$

■

Theorem 7.17: Let X be a random variable. Then for any (Borel) functions $h_1, h_2 : \mathbb{R} \to \mathbb{R}$

$$\boxed{E[h_1(X) \pm h_2(X)] = E[h_1(X)] \pm E[h_2(X)]}$$

proof: Case 1: X is a discrete random variable

Let $h(x) = h_1(x) \pm h_2(x)$

By Theorem 7.6,

$$E[h(X)] = \sum_{x \in X(\Omega)} h(x) p_X(x)$$

$$= \sum_{x \in X(\Omega)} [h_1(x) \pm h_2(x)] p_X(x)$$

$$= \sum_{x \in X(\Omega)} h_1(x) p_X(x) \pm \sum_{x \in X(\Omega)} h_2(x) p_X(x)$$

$$= E[h_1(X)] \pm E[h_2(X)] \quad \text{by Thm 7.6}$$

Case 2: X is continuous random variable

Let $h(x) = h_1(x) + h_2(x)$
By theorem 7.11,

$$E[h(X)] = \int_{-\infty}^{\infty} h(x) f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} (h_1(x) \pm h_2(x))\, f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} h_1(x) f_X(x)\, dx \pm \int_{-\infty}^{\infty} h_2(x) f_X(x)\, dx$$

$$= E[h_1(X)] \pm E[h_2(X)]$$

by Thm 7.11

$\blacksquare$

7.3 Variance

The variance of X is a measure of degree of dispersion of X about its expectation E[X].

Defn 7.18: The variance of a random variable X is

$$\boxed{Var(X) = E[(X - E[X])^2]}$$

whenever these expectations are defined.

The standard deviation of X is then defined to be the positive square root of Var(X)

$$\boxed{sd(X) = \sqrt{Var(X)}}$$

Note: Remember that E[X] and Var(X) are numbers

$$E[X] \in \mathbb{R} \qquad Var(X) \in \mathbb{R}$$

So $E[E[X]] = E[X]$ since expection of a number (here $E[X] \in \mathbb{R}$) is a number.

Theorem: 7.19

$$\boxed{\mathrm{Var}(X) = E[X^2] - (E[X])^2}$$

for any random variable X where variance is defined.

proof: Recall that $E[X]$ is just a number (<u>not</u> a random variable).

Thus $E[E[X]] = E[X]$ and

$$E[X\,E[X]] = E[X]\,E[X] = (E[X])^2$$

↳ linearity of expectations

$$\mathrm{Var}(X) = E[(X - E[X])^2]$$

$$= E[X^2 - 2X\,E[X] + (E[X])^2]$$

a number

$$= E[X^2] - E[2X\,E[X]] + E[\underbrace{(E[X])^2}_{\text{a number}}]$$

By thm 7.17

$$= E[X^2] - 2E[X]^2 + E[X]^2$$

$$= E[X^2] - (E[X])^2 \qquad \blacksquare$$

Example: 7.20 (Example 7.5 continued):

For random variable X, $E[X] = 0$. Thus $X - E[X] = X$

$$Var(X) = E[(X - E[X])^2]$$

$$= E[X^2]$$

$$= (-1)^2 \cdot \frac{1}{2} + 1^2 \cdot \frac{1}{6} + 2^2 \cdot \frac{1}{6} = \frac{4}{3}$$

For random variable T, $E[T] = 1/2$

$$E[T^2] = \left(\frac{1}{2}\right)^2 \cdot \frac{2}{3} + 1^2 \cdot \frac{1}{6} = \frac{1}{3}$$

$$Var(T) = E[T^2] - (E[T])^2$$

$$= \frac{1}{3} - \left(\frac{1}{2}\right)^2$$

$$= \frac{1}{12}$$

Example: 7.21 ***Variance for Poisson Distribution***

Let $X \sim Ber(p)$

$$E[X^2] = \sum_{k=0}^{1} k^2 p_X(k) = p$$

So

$$Var(X) = E[X^2] - (E[X])^2$$

$$= p - p^2$$

$$= p(1-p)$$

So for $\boxed{X \sim Ber(p)}$

$$\boxed{Var(X) = p(1-p)}$$

Example 7.22: **Variance for Exponential Distribution**

If $X \sim Exp(\lambda)$

Using the moment calculated in Example 7.14

$$Var(X) = E[X^2] - (E[X])^2$$

$$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda^2}\right) = \frac{1}{\lambda^2}$$

So for $X \sim Exp(\lambda)$

$$\boxed{Var(X) = \frac{1}{\lambda^2}}$$

Example 7.23 **Variance of Geometric Distribution:**

If $X \sim Geo(p)$. Then writing $q = 1-p$,

$$E[X^2] = \sum_{k=1}^{\infty} k^2 q^{k-1} p$$

$$= \sum_{k=1}^{\infty} k^2 q^{k-1} p + 0$$

$$= \sum_{k=1}^{\infty} k^2 q^{k-1} p + 0^2 q^{k-1} p$$

$$= p \sum_{k=0}^{\infty} k^2 q^{k-1}$$

$$= p \sum_{k=0}^{\infty} \frac{d}{dq}\left(k q^k\right) \qquad \left[\frac{d}{dq} k q^{k-1} = k^2\right]$$

$$= p \frac{d}{dq}\left(\sum_{k=0}^{\infty} k q^k\right) \qquad \left[\text{sum of derivatives is the derivative of sum}\right]$$

$$= p \frac{d}{dq}\left(\sum_{k=0}^{\infty} k q^{k-1} \cdot q \cdot \frac{p}{p}\right)$$

$\frac{p}{p} = 1$ does not change equality

does not depend on k

$$= p \frac{d}{dq}\left(q \cdot \frac{1}{p} \boxed{\sum_{k=0}^{\infty} k q^{k-1} p}\right)$$

$\rightarrow E[x]$ of geometric by Ex 7.3

$$= p\frac{d}{dq}\left(q\cdot\frac{1}{p}\cdot\frac{1}{p}\right)$$

$$= p\frac{d}{dq}\left(\frac{q}{p^2}\right)$$

$$= p\frac{d}{dq}\left(\frac{q}{(1-q)^2}\right)$$

since $q = 1-p$,
$p = 1-q$

$$= p\frac{d}{dq}\left(q(1-q)^{-2}\right)$$

chain rule
+ product rule

$$= p\left[1.(1-q)^{-2} + q(-1)(-2)(1-q)^{-3}\right]$$

$$= p\left[p^{-2} + 2(1-p)p^{-3}\right]$$

$$= \frac{1}{p} + \frac{2.(1-p)}{p^2}$$

$$= \frac{1}{p} + \frac{2}{p^2} - \frac{2}{p} = \frac{2}{p^2} - \frac{1}{p}$$

$$Var(X) = E[X^2] - (E[X])^2$$

$$= \frac{2}{p^2} - \frac{1}{p} - \frac{1}{p^2}$$

$$= \frac{1-p}{p^2}$$

So for $\boxed{X \sim Geo(p)}$

$$\boxed{Var(X) = \frac{1-p}{p^2}}$$

<u>Example 7.24</u>: <u>Variance of normal distribution</u>:

Let $X \sim N(\mu, \sigma^2)$

$$Var(X) = E[(X - E[X])^2] = E[(X-\mu)^2]$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)\,dx$$

$$\int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$$

Using change of variable,

$$Z = \frac{(x-\mu)}{\sigma}$$

$$Var(X) = \int_{-\infty}^{\infty} \sigma^2 Z^2 \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(\frac{-Z^2}{2}\right) \sigma\, dZ$$

$$= \sigma^2 \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Z^2 \exp\left(\frac{-Z^2}{2}\right) dZ$$

$$= -\sigma^2 \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Z \left(\frac{d}{dZ} \exp\left(\frac{-Z^2}{2}\right)\right) dZ$$

$$= \frac{-\sigma^2}{\sqrt{2\pi}} \left(\underbrace{\left[Z \exp\left(\frac{-Z^2}{2}\right) \right]_{-\infty}^{\infty}}_{=0} - \int_{-\infty}^{\infty} \exp\left(\frac{-Z^2}{2}\right) dZ \right)$$

(using integration by parts)

$$= \sigma^2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-Z^2}{2}\right) dZ$$

$$= \sigma^2 \underbrace{\int_{-\infty}^{\infty} \phi(Z)\, dZ}_{=1 \text{ by } (d2)} = \sigma^2$$

So for $\boxed{X \sim N(\mu, \sigma^2)}$

$$\boxed{Var(X) = \sigma^2}$$

Theorem: 7.25

$$\boxed{\mathrm{Var}(aX+b) = a^2\,\mathrm{Var}(X)}$$

for any random variable X with $\mathrm{Var}(X) < \infty$ and any $a, b \in \mathbb{R}$

proof: We use theorems 7.19, 7.16 and 7.17

$$
\begin{aligned}
\mathrm{Var}(aX+b) &= E[(aX+b)^2] - (E[aX+b])^2 \\
&= E[a^2X^2 + 2abX + b^2] - (aE[X]+b)^2 \\
&= a^2E[X^2] + \cancel{2abE[X]} + \cancel{b^2} \\
&\quad - \left(a^2(E[X])^2 + \cancel{2abE[X]} + \cancel{b^2}\right) \\
&= a^2E[X^2] - a^2E[X]^2 \\
&= a^2\left(E[X^2] - E[X]^2\right) \\
&= a^2(\mathrm{Var}(X)) \\
&= a^2\,\mathrm{Var}(X)
\end{aligned}
$$

$\blacksquare$

Example: 7.26 If $X \sim U(0,1)$ then

$$E[X^2] = \int_0^1 \frac{x^2}{b-a} dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

The transformed variable

$$Y = (b-a)X + a$$

is still uniformly distributed

$$Y \sim U(a,b)$$

Thus use thm 7.25 to calculate variance

$$Var[Y] = Var((b-a)X + a)$$

$$= (b-a)^2 Var(X)$$

$$= \frac{(b-a)^2}{12}$$

Example: 7.27 Consider a random variable $X \sim U(-1,1)$ and another random variable Y whose density